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Discrete Applied Mathematics 155 (2007) 1200–1212

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APPLIED
MATHEMATICSwww.elsevier.com/locate/dam

Optimal interleaving schemes for correcting two-dimensional cluster errors[☆]

Wen-Qing Xu^{a, b}, Solomon W. Golomb^c^aDepartment of Mathematics, Capital Normal University, Beijing 100037, China^bDepartment of Mathematics and Statistics, California State University, Long Beach, CA 90840, USA^cDepartment of Electrical Engineering-Systems, University of Southern California, Los Angeles, CA 90089, USA

Received 5 March 2005; received in revised form 22 November 2006; accepted 28 November 2006

Available online 17 January 2007

Abstract

We present an elementary theory of optimal interleaving schemes for correcting cluster errors in two-dimensional digital data. It is assumed that each data page contains a fixed number of, say n , codewords with each codeword consisting of m code symbols and capable of correcting a single random error (or erasure). The goal is to interleave the codewords in the $m \times n$ array such that different symbols from each codeword are separated as much as possible, and consequently, an arbitrary error burst with size up to t can be corrected for the largest possible value of t . We show that, for any given m, n , the maximum possible *interleaving distance*, or equivalently, the largest size of correctable error bursts in an $m \times n$ array, is given by $t = \lfloor \sqrt{2n} \rfloor$ if $n \leq \lceil m^2/2 \rceil$, and $t = m + \lfloor (n - \lceil m^2/2 \rceil)/m \rfloor$ if $n \geq \lceil m^2/2 \rceil$. Furthermore, we develop a simple *cyclic shifting algorithm* that can provide a systematic construction of an $m \times n$ optimal interleaving array for arbitrary m and n . This extends important earlier work on the complementary problem of constructing interleaving arrays that, given the burst size t , minimize the *interleaving degree*, that is, the number of different codewords in a 2-D (or 3-D) array such that any error burst with given size t can be corrected. Our interleaving scheme thus provides the maximum burst error correcting power without requiring prior knowledge of the size or shape of an error burst.

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Keywords: Clusters; Error bursts; ℓ^1 -Distance; Multi-dimensional interleaving; Random error-correcting codes

1. Introduction and statement of main results

Interleaving provides an important means of combating error bursts that often occur in multi-dimensional data handling such as optical recording, holographic storage, charge-coupled devices (CCD), and 2-D bar-codes, etc. The basic idea of interleaving is to mix up the code symbols in the data such that error bursts encountered in the transmission are spread across different codewords as much as possible. Consequently when the data is reconstructed at the receiving end, errors occurring within each codeword become small enough to be correctable by simple random error-correction codes.

The correction of 2-D cluster (or burst) errors has been studied by many authors. Most 2-D burst error correcting codes that have been studied in the literature correct only error bursts of given rectangular [1,5,6,10,12,15,16,21] or other prescribed shapes [2,7,11,20]. See also [17,18] for related interleaving schemes on tori, paths, and cycles, and [4,9,19] for 2-D interleaving schemes for multiple random error correction codes. In [3], the authors have considered

[☆] The material in this paper was presented in part at the IEEE International Symposium on Information Theory, Chicago, IL, June 2004.

E-mail address: wxu@csulb.edu (W.-Q. Xu).

| | | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 0 | 1 | 2 | 3 | 4 | 5 | 0 | 3 | 0 | 3 | 0 | 1 | 2 | 3 | 4 | 5 |
| 0 | 1 | 2 | 3 | 4 | 5 | 1 | 4 | 1 | 4 | 3 | 4 | 5 | 0 | 1 | 2 |
| 0 | 1 | 2 | 3 | 4 | 5 | 2 | 5 | 2 | 5 | 5 | 0 | 1 | 2 | 3 | 4 |

a

b

c

Fig. 1. Examples of 3×6 interleaving arrays. (a) The original un-interleaved 3×6 array. (b) A 3×6 non-optimal interleaving array with interleaving distance 2. (c) A 3×6 optimal interleaving array with interleaving distance 3.

arbitrarily shaped error bursts and studied the problem of interleaving a minimum number of codewords in a 2-D (or 3-D) array such that any error burst of *given* size t can be corrected. By using an elegant *lattice interleaving scheme* that successively shifts each row of the array by b units for a suitably chosen value of b , they have constructed, for any given burst size t , a t -interleaved $n \times n$ square (or infinite 2-D) array that achieves the lowest *interleaving degree* $n = \lceil t^2/2 \rceil$.

Following [3], we consider in this work 2-D error bursts of arbitrary shape and horizontal/vertical connectivity. However, our interest is to study the complementary problem of constructing, for any given 2-D array, interleaving schemes that achieve the maximum possible *interleaving distance*. (The precise definitions are given below.) In other words, instead of fixing the burst size t and minimizing the interleaving degree as in [3], we fix the size of the array (and hence the interleaving degree), and study interleaving schemes that can correct error bursts of size up to t for the largest possible value of t . This results in interleaving arrays that provide the maximum burst error-correcting power without requiring prior knowledge of the size (or shape) of an error burst.

For simplicity, we assume in this paper that all codewords involved have the same length m and are capable of correcting a single random error (or erasure). Let n be the number of codewords contained in the 2-D data, then we have, prior to interleaving, an $m \times n$ rectangular array A_0 with each column of A_0 representing a separate codeword (not necessarily different). For convenience, we may use consecutive symbols $0, 1, 2, \dots$ to distinguish separate codewords and therefore denote $A_0 = (a_{ij})$ with $a_{ij} = j$ for $0 \leq i < m, 0 \leq j < n$. Alternatively, one may define $a_{ij} = S_k, k = i + jm, 0 \leq i < m, 0 \leq j < n, 0 \leq k < mn$ and therefore further distinguish different symbols in each codeword. Then $S_k \approx S_l$, that is, two symbols S_k and S_l belong to the same codeword, if and only if $\lfloor k/m \rfloor = \lfloor l/m \rfloor$. Throughout the paper, we use the notation $\lfloor z \rfloor$ to represent the largest integer less than or equal to z . Similarly, $\lceil z \rceil$ is used to denote the smallest integer greater than or equal to z , that is, $\lceil z \rceil = -\lfloor -z \rfloor$.

Definition 1.1. Let $\mathbf{x}, \mathbf{y} \in \mathbb{Z}^2$, $\mathbf{x} = (x_1, x_2)$, $\mathbf{y} = (y_1, y_2)$. Then the distance between \mathbf{x} and \mathbf{y} is defined as

$$d(\mathbf{x}, \mathbf{y}) = \|\mathbf{x} - \mathbf{y}\|_1 = |x_1 - y_1| + |x_2 - y_2|.$$

Definition 1.2. Two elements $\mathbf{x}, \mathbf{y} \in \mathbb{Z}^2$ are called neighbors (or referred to as connected) if and only if $d(\mathbf{x}, \mathbf{y}) = 1$. A set \mathcal{S} of t elements ($t \geq 2$) in \mathbb{Z}^2 is said to be a *cluster* (or *burst*) of size t if any two elements of \mathcal{S} belong to a sequence of consecutively connected elements contained in \mathcal{S} .

Definition 1.3. Let $A = (a_{ij})$ be an $m \times n$ array whose elements correspond to a permutation of those of the $m \times n$ array A_0 specified above. Then A is called an interleaving of A_0 , and the interleaving distance of A is defined as $\min\{|i_1 - i_2| + |j_1 - j_2| : a_{i_1 j_1} = a_{i_2 j_2}, (i_1, j_1) \neq (i_2, j_2), 0 \leq i_1, i_2 < m, 0 \leq j_1, j_2 < n\}$, that is, the shortest distance between any two same (or equivalent) elements in A . A is called an optimal interleaving array if the interleaving distance of A attains the maximum, that is, there does not exist an interleaving array of A_0 with a larger interleaving distance than that of A .

Example 1.1. Fig. 1(a) shows the original 3×6 array with six different codewords, each occupying a whole column in the array. It can be easily checked that the array in Fig. 1(b) has interleaving distance 2 while the array in Fig. 1(c) has interleaving distance 3. The array in Fig. 1(b) is clearly not optimally interleaved. The array in Fig. 1(c), however, is optimally interleaved, see Theorem 1.1 below.

Notice that for single random error correction codes, an arbitrarily shaped error burst of size t in an interleaved array can be corrected if and only if the array is t -interleaved [3], that is, any cluster of size t contains at most one symbol of

each kind, or equivalently, the ℓ^1 distance between any two same symbols in the array is at least t . Thus, the interleaving distance defined above is the largest value of t such that the array is t -interleaved; and it also gives the largest value of t such that every error cluster of size up to t can be corrected. Given an $m \times n$ array, our goal in this paper may be concisely described as to determine the maximum possible interleaving distance and construct the corresponding optimal interleaving arrays.

The main results of the paper can be summarized as follows.

Theorem 1.1. *For any positive integers m, n , the maximum possible interleaving distance for an $m \times n$ array is given by*

$$T = \begin{cases} \lfloor \sqrt{2n} \rfloor & \text{if } n \leq \lceil m^2/2 \rceil, \\ m + \lfloor (n - \lceil m^2/2 \rceil)/m \rfloor & \text{if } n \geq \lceil m^2/2 \rceil. \end{cases}$$

By using a standard (and a related partial) sphere packing technique, the upper bound in Theorem 1.1 for possible interleaving distances can be established relatively easily. The main difficulty of the proof of Theorem 1.1 is to show that the same upper bound is also exact, that is, the maximum interleaving distance T defined in Theorem 1.1 is indeed achievable. Note that while the lattice interleaving scheme of [3] (see also [14]) can be directly applied to construct an $n \times n$ square optimal interleaving array in the case of $n = \lceil t^2/2 \rceil$ for some integer t , the same approach does not lead to the best interleaving solution for general square (and rectangular) arrays. To overcome the difficulties, we propose a more general, non-uniform *cyclic shifting algorithm* that independently translates each row of the array of codewords by a certain number of places. Although this generally destroys the lattice structure, our cyclic shifting algorithm provides extra flexibility which proves to be sufficient (and also necessary) for the successful construction of optimal interleaving arrays for all values of m and n .

More precisely, our construction of optimal interleaving arrays consists of the following two steps:

Step 1: Choose $\xi_i \in \mathbb{Z}_n = \{0, 1, \dots, n-1\}$ for $0 \leq i < m$ such that for all $i \neq k$, $0 \leq i, k < m$, it holds that

$$|i - k| + |\xi_i - \xi_k| \geq T, \quad |i - k| + n - |\xi_i - \xi_k| \geq T. \quad (1)$$

Step 2: For each $j = 0, 1, \dots, n-1$, place symbol ‘ j ’ at locations $(i, (\xi_i + j) \pmod n)$, where $i = 0, 1, \dots, m-1$, that is,

$$(i, j) \mapsto (i, (\xi_i + j) \pmod n), \quad 0 \leq i < m, \quad 0 \leq j < n. \quad (2)$$

Notice that (2) defines a cyclic translation by ξ_i for row i with $0 \leq i < m$. (For example, for the 3×6 array in Fig. 1(c), we have $\xi_0 = 0$, $\xi_1 = 3$ and $\xi_2 = 1$.) Let $0 \leq i < m$, $0 \leq j < n$, then the symbol a_{ij} at the location (i, j) in the interleaved array $A = (a_{ij})$ is given by

$$a_{ij} = j - \xi_i \pmod n, \quad (3)$$

or equivalently,

$$a_{ij} = S_k, \quad k = i + (j - \xi_i)m \pmod{mn}.$$

Throughout the paper, we follow the usual matrix notation where (i, j) represents the location at row i and column j . Unless otherwise stated, we start with $(0, 0)$ at the upper-left corner.

We now describe the procedures defining ξ_i for $0 \leq i < m$.

Case I: $n \leq \lceil m^2/2 \rceil$. This is the easier case and includes all square arrays. Notice that the first bound in Theorem 1.1 then applies, namely, $T = \lfloor \sqrt{2n} \rfloor$. In this case, we define

$$\xi_i = bi \pmod{n_*}, \quad 0 \leq i < m,$$

where

$$n_* = \lceil t^2/2 \rceil, \quad t = \lfloor \sqrt{2n} \rfloor, \quad b = 2\lceil t/2 \rceil - 1.$$

Case II: $n \geq \lceil m^2/2 \rceil$. This is a very different and much harder case. Note that the maximum interleaving distance $T = \lfloor (n - \lceil m^2/2 \rceil)/m \rfloor + m$ now grows almost linearly with n . Clearly, a different strategy is needed.

In this case, we define first

$$n_* = \lceil \hat{t}^2/2 \rceil, \quad \hat{t} = m, \quad \hat{b} = 2\lceil \hat{t}/2 \rceil - 1,$$

$$\hat{\xi}_i = \hat{b}i \pmod{n_*}, \quad 0 \leq i < m.$$

Then we define

$$\xi_{i_\alpha} = \hat{\xi}_{i_\alpha} + \alpha\eta, \quad 0 \leq \alpha < m,$$

where

$$\eta = \lfloor (n - \lceil m^2/2 \rceil)/m \rfloor$$

and $\{i_0, i_1, \dots, i_{m-1}\}$ is a re-ordering of $\{0, 1, \dots, m-1\}$ such that $\hat{\xi}_{i_0} \leq \hat{\xi}_{i_1} \leq \dots \leq \hat{\xi}_{i_{m-1}}$.

Theorem 1.2. *For any positive integers m, n , the above procedure generates an $m \times n$ optimal interleaving array with maximum interleaving distance T given in Theorem 1.1.*

As an immediate consequence of Theorems 1.1 and 1.2, we observe that the maximum achievable interleaving distance $T = T(m, n)$ is always a monotonic increasing function of n . This implies that the more codewords we use, the larger the size of correctable error bursts we can achieve; and vice versa. In particular, for single-random-error-correcting codewords of length m , if an arbitrary error burst of size up to t is to be corrected, then the minimum number of codewords required is given by $n = \lceil t^2/2 \rceil$ if $t \leq m$, and $n = \lceil m^2/2 \rceil + m(t - m) = mt - \lfloor m^2/2 \rfloor$ if $t \geq m$. Finally we mention that, after the completion of this work, we were able to further extend some of the results presented in this paper to the more general case when the number of codewords in the array is not necessarily equal to the number of rows or columns in the array. We refer to [13] for details.

The rest of the paper is devoted to the proof of Theorems 1.1 and 1.2. After providing the necessary theoretical background in Section 2, we consider first the case of $n \times n$ square arrays ($m = n$) in Section 3. By using a standard sphere packing technique and the optimal interleaving constructions of [3,14] for the special case of $n = \lceil d^2/2 \rceil$ for some integer d , Theorems 1.1 and 1.2 can be proved quite easily. The results are then extended to $m \times n$ rectangular arrays with $n \leq \lceil m^2/2 \rceil$. The remaining case of $n \geq \lceil m^2/2 \rceil$ is studied in Section 4 where a novel partial sphere packing technique is introduced to derive the new bound for the maximum interleaving distance and an interesting re-ordering and successive marching idea is developed for constructing the corresponding optimal interleaving arrays. Section 5 concludes the paper.

2. Theoretical background

2.1. Cyclic shifting algorithm

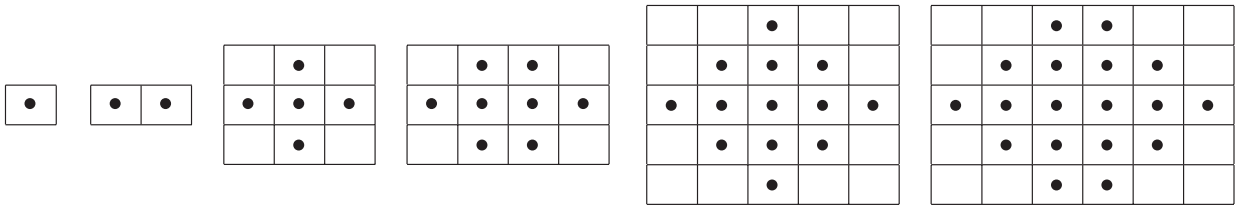
The key part of our construction of optimal interleaving arrays lies in the first step, that is, choosing $\xi_i \in \{0, 1, \dots, n-1\}$, $0 \leq i < m$ such that both inequalities in (1) are satisfied. Note that the first inequality in (1) implies that the distance between any two ‘0’ symbols is at least T . The second inequality in (1) ensures that the same is true for the other symbols.

Lemma 2.1. *Suppose there exist a positive integer t and $\xi_i \in \{0, 1, \dots, n-1\}$, $0 \leq i < m$ such that for all $i \neq k$, $0 \leq i, k < m$, it holds that*

$$|i - k| + |\xi_i - \xi_k| \geq t, \quad |i - k| + n - |\xi_i - \xi_k| \geq t. \quad (4)$$

Then the cyclic shifting algorithm (2) generates an $m \times n$ array with interleaving distance $\geq t$.

Proof. Let $\mathbf{x} = (x_1, x_2)$, $\mathbf{y} = (y_1, y_2)$ be two distinct locations holding the same symbol ‘ j ’ for some $0 \leq j < n$. Then $x_2 = \xi_{x_1} + j \pmod{n}$, $y_2 = \xi_{y_1} + j \pmod{n}$, $0 \leq x_1, y_1 < m$, $x_1 \neq y_1$. Since $-n < x_2 - y_2 < n$ and $x_2 - y_2 = \xi_{x_1} + j \pmod{n} - (\xi_{y_1} + j \pmod{n}) \equiv \xi_{x_1} - \xi_{y_1} \pmod{n}$ it follows that $|x_2 - y_2|$ equals either $|\xi_{x_1} - \xi_{y_1}|$ or $n - |\xi_{x_1} - \xi_{y_1}|$.

Fig. 2. 2-D Spheres $\mathcal{S}_{2,d}$ with $d = 1, 2, 3, 4, 5, 6$.

Then from (4), we obtain $d(\mathbf{x}, \mathbf{y}) = |x_1 - y_1| + |x_2 - y_2| \geq \min\{|x_1 - y_1| + |\xi_{x_1} - \xi_{y_1}|, |x_1 - y_1| + n - |\xi_{x_1} - \xi_{y_1}|\} \geq t$. Lemma 2.1 now follows. \square

The converse to Lemma 2.1 is also true. More precisely, we have the following:

Lemma 2.2. Suppose there exist a positive integer t and $\xi_i \in \{0, 1, \dots, n-1\}$, $0 \leq i < m$ such that the cyclic shifting algorithm (2) generates an $m \times n$ array with interleaving distance t , then (4) holds for all $i \neq k$, $0 \leq i, k < m$.

Proof. The first inequality follows directly from the fact the distance between the two ‘0’ symbols at (i, ξ_i) and (k, ξ_k) is at least t . The second inequality holds trivially if $\xi_i = \xi_k$. (Any two consecutive rows of the array must contain at least one symbol more than once. This implies that the interleaving distance t is always bounded by n .) Without loss of generality, we assume $\xi_i < \xi_k$. Let $j = n - \xi_k$. Then we have $1 \leq j < n$ and $\xi_i + j = n - (\xi_k - \xi_i) < n$. The second inequality now follows as the distance between the two ‘j’ symbols at $(i, n - |\xi_i - \xi_k|)$ and $(k, 0)$ is at least t . \square

2.2. Discrete 2-D spheres

The concept of 2-D and higher-dimensional “spheres” plays a key role in [3,14]. We summarize here some basic properties of 2-D spheres that will be needed later.

Lemma 2.3 (2-D spheres). Let d be a positive integer. Then the following set:

$$\mathcal{S}_{2,d} = \begin{cases} \{\mathbf{x} \in \mathbb{Z}^2 : |x_1| + |x_2| < d/2\} & \text{if } d \text{ is odd,} \\ \{\mathbf{x} \in \mathbb{Z}^2 : |x_1| + |x_2 - 1/2| < d/2\} & \text{if } d \text{ is even} \end{cases}$$

defines a 2-D sphere with diameter d , that is, for any $\mathbf{x} \in \mathcal{S}_{2,d}$, $\mathbf{y} \in \mathcal{S}_{2,d}$, it holds that $d(\mathbf{x}, \mathbf{y}) < d$; and for any $\mathbf{x} \notin \mathcal{S}_{2,d}$, there exists $\mathbf{y} \in \mathcal{S}_{2,d}$ such that $d(\mathbf{x}, \mathbf{y}) \geq d$.

Fig. 2 shows the 2-D spheres $\mathcal{S}_{2,d}$ with $d = 1, 2, 3, 4, 5, 6$. Geometrically, the sphere $\mathcal{S}_{2,d}$ can be constructed recursively by appending all neighbors of $\mathcal{S}_{2,d-2}$, starting with $\mathcal{S}_{2,1} = \{(0, 0)\}$ if d is odd, and $\mathcal{S}_{2,2} = \{(0, 0), (0, 1)\}$ if d is even.

Counting the elements in $\mathcal{S}_{2,d}$, we have [3] (see also Lemma 4.1 below).

Lemma 2.4. For any integer d , it holds that

$$|\mathcal{S}_{2,d}| = \lceil d^2/2 \rceil.$$

It is clear that the sphere $\mathcal{S}_{2,d}$ can be embedded in a $d \times d$ array (or $(d-1) \times d$ array if d is even). For d odd, the sphere $\mathcal{S}_{2,d}$ is centered at $(0, 0)$ and satisfies $\mathbf{x} \in \mathcal{S}_{2,d} \iff -\mathbf{x} \in \mathcal{S}_{2,d}$. For d even, the sphere $\mathcal{S}_{2,d}$ is centered at $(0, 1/2)$ (or between $(0, 0)$ and $(0, 1)$). In this case, we have $\mathbf{x} \in \mathcal{S}_{2,d} \iff -\mathbf{x} + (0, 1) \in \mathcal{S}_{2,d}$.

Let $\mathbf{C} \in \mathbb{Z}^2$. For convenience, we define the following translation of $\mathcal{S}_{2,d}$:

$$\mathcal{S}_{2,d}(\mathbf{C}) = \mathcal{S}_{2,d} + \mathbf{C} = \{\mathbf{x} + \mathbf{C} \in \mathbb{Z}^2 : \mathbf{x} \in \mathcal{S}_{2,d}\}$$

and will also refer to $\mathcal{S}_{2,d}(\mathbf{C})$ as a sphere with diameter d . Then for d odd, we have $\mathbf{x} \in \mathcal{S}_{2,d}(\mathbf{C}) \iff \mathbf{C} \in \mathcal{S}_{2,d}(\mathbf{x})$; and for d even, we have $\mathbf{x} \in \mathcal{S}_{2,d}(\mathbf{C}) \iff \mathbf{C} \in \mathcal{S}_{2,d}(\mathbf{x} - (0, 1))$.

3. Case I: $n \leq |\mathcal{S}_{2,m}|$

3.1. Square arrays

We shall first consider the case of square arrays. In this case, the desired upper bound on the maximum interleaving distance can be easily obtained by using a standard sphere packing technique.

Theorem 3.1. *Let n be a positive integer. Then the maximum interleaving distance for an $n \times n$ array is bounded by $t = \lfloor \sqrt{2n} \rfloor$.*

Proof. We assume $n > 2$ as the case for $n = 1$ or 2 is trivial. Then we have $\sqrt{2n} < n$, and hence $t = \lfloor \sqrt{2n} \rfloor \leq n - 1$. This implies that the sphere $\mathcal{S}_{2,t+1}$ with diameter $t + 1$ can be embedded in the $n \times n$ array. Suppose on the contrary there exists an interleaving for the $n \times n$ array with an interleaving distance $\geq t + 1$. Since the distance between any elements in the sphere $\mathcal{S}_{2,t+1}$ is always less than $t + 1$ (see Lemma 2.3), each element in $\mathcal{S}_{2,t+1}$ has to belong to a different codeword. Therefore, $|\mathcal{S}_{2,t+1}| \leq n$. Using Lemma 2.4, we obtain, $(t + 1)^2/2 \leq |\mathcal{S}_{2,t+1}| \leq n$, that is, $t + 1 \leq \sqrt{2n}$. This contradicts $t = \lfloor \sqrt{2n} \rfloor > \sqrt{2n} - 1$. Theorem 3.1 now follows. \square

The above upper bound is indeed the same as that given in Theorem 1.1. To finish the proof, it remains to show that the interleaving distance $t = \lfloor \sqrt{2n} \rfloor$ can actually be achieved. Note that in the case $n = |\mathcal{S}_{2,d}| = \lceil d^2/2 \rceil$ for some integer d (with $t = \lfloor \sqrt{2n} \rfloor = d$), an $n \times n$ optimal interleaving array can be obtained directly by applying the following important result of [3]:

Theorem 3.2. *Let t be a positive integer and $n = |\mathcal{S}_{2,t}|$. Then a t -interleaved $n \times n$ array (or a t -interleaved infinite 2-D array with interleaving degree n) can be constructed by labeling each element (i, j) , $0 \leq i, j < n$ (or $i, j \in \mathbb{Z}$) by integer $j - bi \pmod{n}$ where $b = t$ if t is odd, and $b = t - 1$ if t is even.*

The above construction of [3] actually defines a class of 2-D *lattice interleavers* [3]. The key idea is to successively (and cyclically) shift each row of the array by b units for a suitably chosen value of b . This corresponds to the cyclic shifting scheme $(i, j) \mapsto (i, (j + \xi_i) \pmod{n})$ with $\xi_i = bi \pmod{n}$. Note that in [3], the authors use $b = t + 1$ when t is even. It can be shown that $b = t - 1$ also works and is the smallest number that works in this case. Furthermore, by symmetry, one may also choose $b = -t$ if t is odd, and $b = -(t \pm 1)$ if t is even.

However, for general $n \times n$ square arrays with $n \neq |\mathcal{S}_{2,t}|$, the same approach may no longer lead to an optimal $n \times n$ interleaving array (consider, for example, the case of a 9×9 array). It turns out that in those cases, the more general *non-uniform* cyclic shifting algorithm will be needed.

To proceed, we first observe that, by using Lemmas 2.1 and 2.2 (and restricting to $m \times n$ sub-arrays for arbitrary m), it is easy to see that Theorem 3.2 is actually equivalent to the following:

Lemma 3.1. *Let $n = |\mathcal{S}_{2,t}|$ for some t and define*

$$b = 2\lceil t/2 \rceil - 1, \quad \xi_i = bi \pmod{n}, \quad i \in \mathbb{Z}. \quad (5)$$

Then for all $i \neq k$, $i, k \in \mathbb{Z}$, it holds that

$$|i - k| + |\xi_i - \xi_k| \geq t, \quad |i - k| + n - |\xi_i - \xi_k| \geq t. \quad (6)$$

Remark 3.1. Lemma 3.1 can also be proved directly. For t odd, this can be found in the proof of Theorem 1 in [14].

It is interesting to note that while the construction of a t -interleaved $n \times n$ array requires only the existence of $\xi_i \in \{0, 1, \dots, n - 1\}$ for $0 \leq i < n$ such that (6) is satisfied for all $0 \leq i, k < n$ with $i \neq k$, Lemma 3.1 shows that in the case of $n = |\mathcal{S}_{2,t}|$, one can actually define ξ_i for all $i \in \mathbb{Z}$ which satisfies (6) for all $i, k \in \mathbb{Z}$ with $i \neq k$. This fact has the following important consequence:

| | | | | |
|---|---|---|---|---|
| 0 | 1 | 2 | 3 | 4 |
| 2 | 3 | 4 | 0 | 1 |
| 4 | 0 | 1 | 2 | 3 |
| 1 | 2 | 3 | 4 | 0 |
| 3 | 4 | 0 | 1 | 2 |

Fig. 3. The 5×5 Latin square optimal interleaving array generated by (9).

| | | | | | |
|---|---|---|---|---|---|
| 0 | 1 | 2 | 3 | 4 | 5 |
| 3 | 4 | 5 | 0 | 1 | 2 |
| 5 | 0 | 1 | 2 | 3 | 4 |
| 2 | 3 | 4 | 5 | 0 | 1 |
| 4 | 5 | 0 | 1 | 2 | 3 |
| 0 | 1 | 2 | 3 | 4 | 5 |

a

| | | | | | |
|---|---|---|---|---|---|
| 0 | 1 | 2 | 3 | 4 | 5 |
| 3 | 4 | 5 | 0 | 1 | 2 |
| 5 | 0 | 1 | 2 | 3 | 4 |
| 2 | 3 | 4 | 5 | 0 | 1 |
| 4 | 5 | 0 | 1 | 2 | 3 |
| 1 | 2 | 3 | 4 | 5 | 0 |

b

| | | | | | |
|---|---|---|---|---|---|
| 0 | 1 | 2 | 3 | 4 | 5 |
| 2 | 3 | 4 | 5 | 0 | 1 |
| 4 | 5 | 0 | 1 | 2 | 3 |
| 1 | 2 | 3 | 4 | 5 | 0 |
| 3 | 4 | 5 | 0 | 1 | 2 |
| 5 | 0 | 1 | 2 | 3 | 4 |

c

Fig. 4. (a) The 6×6 non-Latin square optimal interleaving array generated by (9). (b)–(c) 6×6 Latin square optimal interleaving arrays.

Lemma 3.2. Let n be an arbitrary positive integer and define

$$t = \lfloor \sqrt{2n} \rfloor, \quad n_* = |\mathcal{S}_{2,t}|, \quad b = 2\lceil t/2 \rceil - 1, \quad (7)$$

$$\xi_i = bi \pmod{n_*}, \quad 0 \leq i < n. \quad (8)$$

Then we have for all $i \neq k$, $0 \leq i, k < n$:

$$|i - k| + |\xi_i - \xi_k| \geq t, \quad |i - k| + n - |\xi_i - \xi_k| \geq t.$$

Proof. Note that the definition of t and n_* in (7) implies $n_* \leq n$. Lemma 3.2 then follows immediately from Lemma 3.1. \square

Combining Lemmas 3.2, 2.1, and Theorem 3.1, we now obtain

Theorem 3.3. Let n be a positive integer and define ξ_i as in Lemma 3.2. Then the cyclic shifting algorithm

$$(i, j) \mapsto (i, (j + \xi_i) \pmod{n}), \quad 0 \leq i, j < n \quad (9)$$

generates an $n \times n$ optimal interleaving array with maximum interleaving distance $t = \lfloor \sqrt{2n} \rfloor$.

Remark 3.2. In the case $n = |\mathcal{S}_{2,t}|$, we have $bt = 2n - 1 \equiv -1 \pmod{n}$ for t odd, and $b(t + 1) = 2n - 1 \equiv -1 \pmod{n}$ for t even. This implies $\gcd(b, n) = 1$ and hence $\xi_i \neq \xi_k$ for all $0 \leq i, k < n$, $i \neq k$. It follows that the optimal $n \times n$ interleaving array generated by the scheme (9) is always a *Latin square* [8] in which each row and each column contains each symbol exactly once. This provides additional error correcting power to the array in that all linear error bursts occupying a whole row or column can also be corrected. See Fig. 3 for the case $n = 5$. This is, however, no longer the case when $n \neq |\mathcal{S}_{2,t}|$ for $t = \lfloor \sqrt{2n} \rfloor$. Note that in such cases, we have, $n > n_* = |\mathcal{S}_{2,t}|$, and hence by (8), $\xi_0 = \xi_{n_*} = 0$. Nevertheless, it is still possible [22] to construct Latin square optimal interleaving arrays, see Fig. 4 for an example.

3.2. Rectangular arrays: $n \leq |\mathcal{S}_{2,m}|$

Next we note that with slight change in notation, the same analysis as in the above can also be used to obtain

Theorem 3.4. Let m, n be two positive integers and define

$$t = \lfloor \sqrt{2n} \rfloor, \quad n_* = |\mathcal{S}_{2,t}|, \quad b = 2\lceil t/2 \rceil - 1, \quad (10)$$

$$\xi_i = bi \pmod{n_*}, \quad 0 \leq i < m. \quad (11)$$

Then the following cyclic shifting interleaving scheme:

$$(i, j) \mapsto (i, (\xi_i + j) \pmod{n}), \quad 0 \leq i < m, \quad 0 \leq j < n \quad (12)$$

generates an $m \times n$ array with interleaving distance $\geq t = \lfloor \sqrt{2n} \rfloor$.

On the other hand, similar to Theorem 3.1, we have

Theorem 3.5. Assume that

$$n < \begin{cases} (m+1)^2/2 & \text{if } m \text{ is odd,} \\ m^2/2 & \text{if } m \text{ is even.} \end{cases} \quad (13)$$

Then the maximum interleaving distance for the $m \times n$ array is bounded by $t = \lfloor \sqrt{2n} \rfloor$.

Proof. Let $t = \lfloor \sqrt{2n} \rfloor$. Then it can be easily checked that assumption (13) is equivalent to $m \geq t$ if t is odd, and $m \geq t+1$ if t is even. This ensures that the 2-D sphere $\mathcal{S}_{2,t+1}$ can be embedded in the given $m \times n$ rectangular array (except the trivial case $n = 2$). With slight modifications, the same proof of Theorem 3.1 now applies. \square

Remark 3.3. Assumption (13) covers all cases $n \leq |\mathcal{S}_{2,m}|$ except $n = |\mathcal{S}_{2,m}|$ for m even. In that case, the maximum interleaving distance will be shown in Theorem 4.1 to be again bounded by $t = \lfloor \sqrt{2n} \rfloor = m$.

As a corollary, we now obtain

Theorem 3.6. Let $n \leq |\mathcal{S}_{2,m}|$ and define ξ_i as in (10), (11). Then the interleaving scheme (12) generates an $m \times n$ optimal interleaving array with maximum interleaving distance $t = \lfloor \sqrt{2n} \rfloor$.

4. Case II: $n \geq |\mathcal{S}_{2,m}|$

Finally, we consider the case $n \geq |\mathcal{S}_{2,m}|$. Note that in this case, the 2-D sphere $\mathcal{S}_{2,t+1}$ with $t = \lfloor \sqrt{2n} \rfloor$ may no longer be embedded in the $m \times n$ array, and the previous sphere packing argument no longer applies. While the same interleaving (12) still generates an $m \times n$ array with an interleaving distance $\geq t = \lfloor \sqrt{2n} \rfloor$, see Theorem 3.4, the result may no longer be optimal. The following examples show that in such cases it is indeed possible to construct interleaving arrays with interleaving distances $> t = \lfloor \sqrt{2n} \rfloor$.

Example 4.1. For $m = 2$, the concepts of interleaving distance and optimal interleaving arrays remain well defined (for erasure correcting codewords). Let $n = 4$, the above interleaving scheme (12) then generates a 2×4 array with interleaving distance $\lfloor \sqrt{2n} \rfloor = 2$, see Fig. 5(a). However, the maximum possible interleaving distance for 2×4 arrays is actually given by $T = 1 + \lfloor n/2 \rfloor = 3$ and an optimal 2×4 interleaving array can be obtained by choosing $\xi_0 = 0$ and $\xi_1 = 2$, see Fig. 5(b).

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 0 | 1 | 2 | 3 | 0 | 1 | 2 | 3 |
| 3 | 0 | 1 | 2 | 2 | 3 | 0 | 1 |
| a | | | | b | | | |

Fig. 5. (a) The 2×4 array generated by (12) with interleaving distance $\lfloor \sqrt{2n} \rfloor = 2$. (b) A 2×4 optimal interleaving array with maximum interleaving distance $1 + \lfloor n/2 \rfloor = 3$.

| | | | | | | | | | | | | | | | |
|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|
| 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 |
| 11 | 12 | 13 | 14 | 15 | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
| 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 0 | 1 | 2 | 3 | 4 | 5 |
| 14 | 15 | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 |

a

| | | | | | | | | | | | | | | | |
|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|
| 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 |
| 9 | 10 | 11 | 12 | 13 | 14 | 15 | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
| 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 0 | 1 | 2 | 3 |
| 13 | 14 | 15 | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 |

b

Fig. 6. (a) The 4×16 interleaving array generated by (12) with interleaving distance $\lfloor \sqrt{2n} \rfloor = 5$ and $\xi = (0, 5, 10, 2)$. (b) A 4×16 array with (maximum) interleaving distance 6 and $\xi = (0, 7, 12, 3)$.

Remark 4.1. In general, for $2 \times n$ arrays, the maximum interleaving distance is bounded by $\lfloor n/2 \rfloor + 1$ as the distance between the $(1, \lfloor n/2 \rfloor)$ element and any other element in the array is bounded by

$$\max\{d((1, \lfloor n/2 \rfloor), (0, 0)), d((1, \lfloor n/2 \rfloor), (0, n-1))\} = \lfloor n/2 \rfloor + 1.$$

This bound can be achieved by taking $\xi_0 = 0$ and $\xi_1 = \lfloor n/2 \rfloor$.

Example 4.2. For 4×16 arrays, we have $\lfloor \sqrt{2n} \rfloor = 5$. The cyclic shifting construction (12) now gives an array with interleaving distance $\lfloor \sqrt{2n} \rfloor = 5$ with $\xi = (0, 5, 10, 2)$, see Fig. 6(a). However, it can be checked that Fig. 6(b) provides a 4×16 array with (maximum) interleaving distance $6 > \lfloor \sqrt{2n} \rfloor = 5$, where $\xi = (0, 7, 12, 3)$.

To overcome these difficulties, we now introduce a novel partial sphere packing technique similar to that used in the proof of Theorem 3.1 and derive the new upper bound for the maximum interleaving distance for general $m \times n$ arrays with $n \geq |\mathcal{S}_{2,m}|$.

Lemma 4.1. Let $d > m$. Then the number of elements in the center m rows of the sphere $\mathcal{S}_{2,d}$ is given by $N = md + |\mathcal{S}_{2,m}| - m^2$.

Proof. First we note that the assumption $d > m$ implies that there are at least m rows in the sphere $\mathcal{S}_{2,d}$. For m odd, the center m rows of $\mathcal{S}_{2,d}$ correspond to $x_1 = 0, \pm 1, \dots$, and $\pm(m-1)/2$, see Lemma 2.3. Therefore, we have in this case

$$\begin{aligned} N &= d + 2(d-2) + 2(d-4) + \dots + 2(d-(m-1)) \\ &= md - 2(2+4+\dots+(m-1)) \\ &= md - (m^2-1)/2 = md + |\mathcal{S}_{2,m}| - m^2. \end{aligned}$$

Similarly, for m even, the center m rows of $\mathcal{S}_{2,d}$ then correspond to $x_1 = 0, \pm 1, \dots, \pm(m-2)/2$, and $m/2$ (or $-m/2$, but not both). Then

$$\begin{aligned} N &= d + 2(d-2) + \dots + 2(d-(m-2)) + (d-m) \\ &= md - 2(2+4+\dots+(m-2)) - m \\ &= md - m^2/2 = md + |\mathcal{S}_{2,m}| - m^2. \end{aligned}$$

Lemma 4.1 now follows. \square

Theorem 4.1. Let $n \geq |\mathcal{S}_{2,m}|$. Then the maximum interleaving distance for an $m \times n$ array is bounded by $t = m + \lfloor (n - |\mathcal{S}_{2,m}|)/m \rfloor$.

Proof. Let $d = t + 1 = m + 1 + \lfloor (n - |\mathcal{S}_{2,m}|)/m \rfloor$. Then the assumption $n \geq |\mathcal{S}_{2,m}|$ implies $d > m$. Next we show that it is also true that $d \leq n$ (except for the trivial case $m = n = 2$). This is because for $m = 2$, we have $d = 2 + \lfloor n/2 \rfloor \leq n$ if $n \geq 3$; and for $m \geq 3$, we have $|\mathcal{S}_{2,m}| > m + 1$ and hence $n = |\mathcal{S}_{2,m}| + (n - |\mathcal{S}_{2,m}|) > m + 1 + m \lfloor (n - |\mathcal{S}_{2,m}|)/m \rfloor \geq d$. This ensures that the center m rows of the 2-D sphere $\mathcal{S}_{2,d}$ can be embedded in the $m \times n$ array.

Suppose there exists an $m \times n$ array with interleaving distance $\geq d$. Since the center m rows of the sphere $\mathcal{S}_{2,d}$ can be embedded in the $m \times n$ array and the distance between any two elements in this set is less than d , each of these N elements then has to represent a different codeword. This forces $n \geq N$. Using Lemma 4.1, we now have, $md \leq m^2 + n - |\mathcal{S}_{2,m}|$, that is, $d \leq m + \lfloor (n - |\mathcal{S}_{2,m}|)/m \rfloor$. This, however, contradicts with the definition of d above. Theorem 4.1 now follows. \square

The above estimate in Theorem 4.1 actually gives the exact upper bound for the maximum interleaving distance for an $m \times n$ array in the case $n \geq |\mathcal{S}_{2,m}|$, see Theorem 1.1. See also Remark 4.1 (and Example 4.1) for the special case of $m = 2$.

Theorem 4.1 implies that for $n \geq |\mathcal{S}_{2,m}|$, the $m \times n$ interleaving array constructed in Theorem 3.4 is in general non-optimal. This is because for $n \geq |\mathcal{S}_{2,m}|$, it holds that $m + \lfloor (n - |\mathcal{S}_{2,m}|)/m \rfloor \geq \lfloor \sqrt{2n} \rfloor$. To see this, we define $\eta = \lfloor (n - |\mathcal{S}_{2,m}|)/m \rfloor$, $\eta \geq 0$. Then we have $n < |\mathcal{S}_{2,m}| + m(\eta + 1) \leq (m^2 + 1)/2 + m(\eta + 1) \leq (m + \eta + 1)^2/2$, and hence $\lfloor \sqrt{2n} \rfloor \leq m + \eta = m + \lfloor (n - |\mathcal{S}_{2,m}|)/m \rfloor$.

However, for $\eta = 0$, that is, $|\mathcal{S}_{2,m}| \leq n < |\mathcal{S}_{2,m}| + m$, we do have $m + \lfloor (n - |\mathcal{S}_{2,m}|)/m \rfloor = m = \lfloor \sqrt{2n} \rfloor$. This, together with Theorem 3.4, then implies

Theorem 4.2. Let $|\mathcal{S}_{2,m}| \leq n < |\mathcal{S}_{2,m}| + m$. Then the same interleaving scheme in Theorem 3.4 generates an $m \times n$ optimal interleaving array with maximum interleaving distance m .

Given an $m \times n$ rectangular array with interleaving distance t , our next result provides an important bootstrapping method to construct an $m \times (n + m)$ array with an interleaving distance $\geq t + 1$.

Lemma 4.2. Let m, \hat{n} be two positive integers. Assume that there exist $\hat{t} > 0$ and $\hat{\xi}_i \in \{0, 1, \dots, \hat{n} - 1\}$, $i = 0, 1, \dots, m - 1$ such that for all $i \neq k$, $0 \leq i, k < m$, it holds that

$$|i - k| + |\hat{\xi}_i - \hat{\xi}_k| \geq \hat{t}, \quad |i - k| + \hat{n} - |\hat{\xi}_i - \hat{\xi}_k| \geq \hat{t}, \quad (14)$$

that is, the following interleaving scheme:

$$(i, j) \mapsto (i, (\hat{\xi}_i + j) \pmod{\hat{n}}), \quad 0 \leq i < m, \quad 0 \leq j < \hat{n}$$

generates an $m \times \hat{n}$ array with interleaving distance $\geq \hat{t}$.

Let $\{i_0, i_1, \dots, i_{m-1}\}$ be a re-ordering of $\{0, 1, \dots, m - 1\}$ such that

$$\hat{\xi}_{i_0} \leq \hat{\xi}_{i_1} \leq \dots \leq \hat{\xi}_{i_{m-1}},$$

and define,

$$n = \hat{n} + m, \quad \xi_{i_\alpha} = \hat{\xi}_{i_\alpha} + \alpha, \quad 0 \leq \alpha < m,$$

then the following scheme:

$$(i, j) \mapsto (i, (\xi_i + j) \pmod{n}), \quad 0 \leq i < m, \quad 0 \leq j < n$$

generates an $m \times n$ array with interleaving distance $\geq t = \hat{t} + 1$.

Proof. First it is clear that $\xi_{i_\alpha} \geq 0$ and $\xi_{i_\alpha} = \hat{\xi}_{i_\alpha} + \alpha < \hat{n} + m = n$ for all $0 \leq \alpha < m$. By Lemma 2.1, we now only have to show that for all $i \neq k$, $0 \leq i, k < m$ it holds that

$$|i - k| + |\xi_i - \xi_k| \geq t, \quad |i - k| + n - |\xi_i - \xi_k| \geq t.$$

| | | | | | | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|----|----|---|----|----|---|---|----|----|---|----|----|
| 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 |
| 5 | 6 | 7 | 0 | 1 | 2 | 3 | 4 | 7 | 8 | 9 | 10 | 11 | 0 | 1 | 2 | 3 | 4 | 5 | 6 |
| 2 | 3 | 4 | 5 | 6 | 7 | 0 | 1 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 0 | 1 | 2 |
| 7 | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 10 | 11 | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |

a

b

Fig. 7. (a) A 4×8 array with (maximum) interleaving distance 4 and $\xi = (0, 3, 6, 1)$. (b) The five-interleaved 4×12 array generated by Lemma 4.2 with $\xi = (0, 5, 9, 2)$.

Let $\alpha, \beta \in \{0, 1, \dots, m-1\}$, $\alpha \neq \beta$. Then as a result of the re-ordering, we have

$$|\xi_{i_\alpha} - \xi_{i_\beta}| = |\hat{\xi}_{i_\alpha} - \hat{\xi}_{i_\beta} + \alpha - \beta| = |\hat{\xi}_{i_\alpha} - \hat{\xi}_{i_\beta}| + |\alpha - \beta|.$$

Therefore by (14), we have

$$|i_\alpha - i_\beta| + |\xi_{i_\alpha} - \xi_{i_\beta}| = |i_\alpha - i_\beta| + |\hat{\xi}_{i_\alpha} - \hat{\xi}_{i_\beta}| + |\alpha - \beta| \geq \hat{t} + 1 = t,$$

and

$$\begin{aligned} |i_\alpha - i_\beta| + n - |\xi_{i_\alpha} - \xi_{i_\beta}| &= |i_\alpha - i_\beta| + \hat{n} + m - |\hat{\xi}_{i_\alpha} - \hat{\xi}_{i_\beta}| - |\alpha - \beta| \\ &= |i_\alpha - i_\beta| + \hat{n} - |\hat{\xi}_{i_\alpha} - \hat{\xi}_{i_\beta}| + (m - |\alpha - \beta|) \\ &\geq \hat{t} + 1 = t. \end{aligned}$$

Lemma 4.2 now follows easily. \square

Example 4.3 (4×8 and 4×12 arrays). Let $m = 4$, $\hat{n} = 8$ and $n = 12$. In Fig. 7(a), we have a 4×8 array with (maximum) interleaving distance 4 and $\hat{\xi} = (0, 3, 6, 1)$. Fig. 7(b) shows the 5-interleaved 4×12 array generated by the procedures in Lemma 4.2 with $\xi = (0, 5, 9, 2)$. Note that one can further apply Lemma 4.2 to Fig. 7(b) to obtain the 6-interleaved 4×16 array in Fig. 6(c) where $\xi = (0, 7, 12, 3)$.

With Theorem 4.2 and Lemma 4.2, we are now ready to prove

Theorem 4.3. Consider a general $m \times n$ rectangular array with $n \geq |\mathcal{S}_{2,m}|$. Define

$$\begin{aligned} \hat{n} &= n - \eta m, \quad \eta = \lfloor (n - |\mathcal{S}_{2,m}|)/m \rfloor, \\ n_* &= |\mathcal{S}_{2,m}|, \quad \hat{b} = 2\lceil m/2 \rceil - 1, \\ \hat{\xi}_i &= \hat{b}i \pmod{n_*}, \quad 0 \leq i < m. \end{aligned} \tag{15}$$

Next we re-order $\hat{\xi}_0, \hat{\xi}_1, \dots, \hat{\xi}_{m-1}$ such that

$$\hat{\xi}_{i_0} \leq \hat{\xi}_{i_1} \leq \dots \leq \hat{\xi}_{i_{m-1}}$$

and define,

$$\xi_{i_\alpha} = \hat{\xi}_{i_\alpha} + \alpha\eta, \quad 0 \leq \alpha < m,$$

then the following scheme:

$$(i, j) \mapsto (i, (\xi_i + j) \pmod{n}), \quad 0 \leq i < m, \quad 0 \leq j < n$$

generates an $m \times n$ array with the maximum interleaving distance $T = m + \lfloor (n - |\mathcal{S}_{2,m}|)/m \rfloor$.

Proof. Note that with \hat{n} defined in (15), we have

$$\hat{n} \equiv n \pmod{m}, \quad |\mathcal{S}_{2,m}| \leq \hat{n} < |\mathcal{S}_{2,m}| + m.$$

Therefore by Theorem 4.2, the following interleaving scheme:

$$(i, j) \mapsto (i, (\hat{\xi}_i + j) \pmod{\hat{n}}), \quad 0 \leq i < m, \quad 0 \leq j < \hat{n}$$

generates an $m \times \hat{n}$ array with maximum interleaving distance $= \lfloor \sqrt{2\hat{n}} \rfloor = m$. Then by repeatedly applying Lemma 4.2, it follows that the following interleaving scheme:

$$(i, j) \mapsto (i, (\xi_i + j) \pmod{n}), \quad 0 \leq i < m, \quad 0 \leq j < n$$

then generates an $m \times n$ interleaving array with an interleaving distance $\geq m + \eta = m + \lfloor (n - \lfloor \mathcal{S}_{2,m} \rfloor) / m \rfloor$. This, together with Theorem 4.1, completes the proof of Theorem 4.3. \square

5. Conclusions

In this paper, we have presented optimal interleaving schemes for correcting arbitrarily shaped error bursts in 2-D digital data. Given any $m \times n$ array of n single-error-correcting codewords, our results show that the maximum size of correctable error bursts is given by $t = \lfloor \sqrt{2n} \rfloor$ for $n \leq \lceil m^2/2 \rceil$ and $t = m + \lfloor (n - \lceil m^2/2 \rceil) / m \rfloor$ for $n \geq \lceil m^2/2 \rceil$, and the maximum burst error correcting power can be achieved by cyclically shifting each row of the array of codewords by a certain number of places, where the codewords form the columns of the array. These interleaving schemes provide the maximum burst error correcting power without requiring prior knowledge of the size or shape of an error burst. In particular, our cyclic shifting algorithm outperforms the lattice interleaving schemes by providing a systematic construction of an $m \times n$ optimal interleaving array for all values of m and n .

Acknowledgments

The authors would like to thank the anonymous reviewers for their very helpful comments. The first author also wishes to thank Dr. Ximin Zhang for a number of useful discussions related to the paper.

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